(1) $y' + y = 0$; $y = f(x)$

1° $f(x) = e^{-x}$

$f'(x) = e^{-x} \cdot (-1) = -e^{-x}$

$-e^{-x} + e^{-x} = 0$ ✓

$f(x) = e^{-x}$ ~~napr~~ particular solution

2° $f(x) = 2 \cdot e^{-x}$

$f'(x) = 2 \cdot e^{-x} \cdot (-1) = -2e^{-x}$

$y' + y = -2e^{-x} + 2e^{-x} = 0$ ✓

particular solution

3° $f(x) = c \cdot e^{-x}$

$f'(x) = c \cdot e^{-x} \cdot (-1) = -c \cdot e^{-x}$

$y' + y = -c \cdot e^{-x} + c \cdot e^{-x} = 0$ ✓

$f(x) = c \cdot e^{-x}$ general solution

$c = 1$ 1° ✓

$c = 2$ 2° ✓

- $y' + y = 0$

  first order equation ; $y = c \cdot e^{-x}$

- $y'' - 2y' + 3y = 6x$

  second order equation

  $\boxed{y = f(x, c_1, c_2)}$ general solution

- $y''' - y' + x^2 = 0$

  $y = f(x, c_1, c_2, c_3)$

$$y = f(x, C_1, C_2, C_3)$$

* $F(x, y, y', y'', \dots, y^{(n)}) = 0$

$$y = f(x, C_1, C_2, \dots, C_n)$$

general solution

* Integration and diff. eq. ?!

$$\int e^{-x}dx = -e^{-x} + C$$

$$F(x, C) = -e^{-x} + C \checkmark$$

$$F_x'(x, C) = e^{-x} \checkmark$$

$$y' = e^{-x}$$

differential equations $\rightleftarrows$ economy model's

$$\left\{ \begin{array}{lcl} \text{growth rate} & \to & \text{први извод} \\ \text{trend} & \to & \text{други извод} \end{array} \right\}$$

<u>Malthus model of growth popolaction</u>

$y(t)$ – size of a population of a species of animal at time t

$t = 0$; $y(t)$ – size in thousands;

- Population has constant growth rate r.

$$\frac{dy}{dt} = r \cdot y;$$

rate at which population grows is proportional to the size of population

- Activity 5.1 Show that $y(0)e^{rt}$ solves

• Activity 5.1 Show that $y(0)e^{rt}$ solves Malthus equation $dy/dt = r \cdot y$

$*$ $y = y(0) \cdot e^{rt}$

$y' = y(0) \cdot e^{r \cdot t} \cdot r$

$$\frac{dy}{dt} = r \cdot y$$

$$y(0) \cdot e^{rt} \cdot r = r \cdot y(0) \cdot e^{r \cdot t} \quad \checkmark$$

∘ $y = y(0) \cdot e^{rt} \xrightarrow{t \to \infty} +\infty$

•• Logistic model

$*$ growth rate $\frac{dy/dt}{y}$ is not constant, but is a decreasing function of y.

$$\Downarrow \quad \boxed{a - by}\ ; \ a, b > 0$$

$$\frac{dy/dt}{y} = a - by$$

~~5.~~ Separable equations

$$\frac{dy}{dx} = f(y) \cdot g(x)$$

∘ $\frac{dy}{dx} = y \cdot e^{x} \quad \checkmark$

∘ $\frac{dy}{dx} = x^3 y + x y^3 \quad ??$

Solving: $\frac{dy}{dx} = f(y) \cdot g(x)$

$$\frac{dy}{f(y)} = g(x) \cdot dx$$

$$\frac{dy}{f(y)} = g(x)\cdot dx$$

$$\int \frac{dy}{f(y)} = \int g(x)\,dx$$

EXAMPLE

$$\frac{dy}{dx} = -\frac{x^3y^2}{\sqrt{1+x^4}}$$

$$\frac{1}{y^2}\,dy = -\frac{x^3}{\sqrt{1+x^4}}\,dx \qquad \sqrt{1+x^4} = t$$

$$\int \frac{1}{y^2}\,dy = \int -\frac{x^3}{\sqrt{1+x^4}}\,dx \qquad \frac{1}{2\sqrt{1+x^4}}\cdot 4x^3dx = dt$$

$$\frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{2}\,dt$$

$$-\frac{1}{y} = -\int \frac{1}{2}\,dt$$

$$-\frac{1}{y} = -\frac{1}{2}t + c$$

$$-\frac{1}{y} = -\frac{1}{2}\cdot\sqrt{1+x^4} + c; \qquad \boxed{y = ?}$$

$$\frac{1}{y} = \frac{1}{2}\sqrt{1+x^4} - c$$

$$y = \frac{1}{\frac{1}{2}\sqrt{1+x^4} - c} \quad \begin{matrix}/2\\ /2\end{matrix}$$

$$y = \frac{2}{\sqrt{1+x^4} - 2c} \qquad -2c = k$$

$$y = \frac{2}{\sqrt{1+x^4} + k}$$ - general solution;

* $y(0) = 1$ - particular solution;

$$y(0) = \frac{2}{1+k} = 1 \quad \Rightarrow \quad k = 1$$

$$y(x) = \frac{2}{\sqrt{1+x^4} + 1};$$

$$y(x) = \frac{x}{\sqrt{1+x^4}+1};$$

- $$\frac{dy}{dt} = r \cdot y$$

$$\frac{dy}{y} = r \cdot dt$$

$$\int \frac{dy}{y} = \int r \cdot dt$$

$$\ln y = r \cdot t + c$$

$$y = e^{rt+c} = e^{c} \cdot e^{rt}$$

$$y(0) = e^{c} \cdot e^{0} = e^{c}$$

$$y = y(0) \cdot e^{rt};$$ $\boxed{\begin{array}{l} t \to \infty \\ y \to +\infty \end{array}}$

* logistic equation;

$$\frac{\frac{dy}{dt}}{y} = \underline{\underline{a}} - \underline{\underline{b}} y$$

$$\frac{dy}{dt} = y \cdot (a - by) \quad ; \left\{ \begin{array}{l} a = 1000 \\ \underline{b = 2} \end{array} \right\}$$

$$\frac{dy}{dt} = y \cdot (100 - 2y)$$ separable equation;

$$\frac{dy}{(100-2y) \cdot y} = dt$$

$$\int \frac{dy}{\ } = \ = \int dt = t + c$$

$$\int \frac{dy}{(1000-2y)\cdot y} = \int dt = t + C$$

Partial fractions!

$$\frac{1}{(1000-2y)y} = \frac{A}{1000-2y} + \frac{B}{y} \quad / \, y\cdot(1000-2y)$$

$$* \quad 1 = A\cdot y + B(1000-2y)$$

$y=0$: $\quad 1 = B\cdot 1000 \Rightarrow B = \frac{1}{1000}$

$y=500$: $\quad 1 = A\cdot 500 \Rightarrow A = \frac{1}{500}$

$$\int \frac{1}{(1000-2y)y}\cdot dy = \int \frac{\frac{1}{500}}{1000-2y}\,dy + \int \frac{\frac{1}{1000}}{y}\,dy$$

$$= \frac{1}{500}\ln(1000-2y)\cdot\frac{-1}{2} + \frac{1}{1000}\ln y$$

$$= \frac{1}{1000}\left(\ln y - \ln(1000-2y)\right)$$

$$= \frac{1}{1000}\ln\frac{y}{1000-2y}$$

$$\rightarrow \quad \frac{1}{1000}\ln\frac{y}{1000-2y} = t + C \quad \checkmark$$

$$\ln\frac{y}{1000-2y} = 1000(t+C)$$

$$\frac{y}{1000-2y} = e^{1000(t+C)}$$

$$\frac{1000-2y}{y} = e^{-1000(t+C)}$$

$$\frac{1000}{y} - 2 = e^{-1000(t+C)}$$

$$\frac{1000}{y} = 2 + e^{-1000(t+C)}$$

$$\frac{y}{1000} = \frac{1}{2+e^{-1000(t+C)}}$$

$$\frac{y}{1000} = \frac{\quad}{2+e^{-1000(t+c)}}$$

$$y = \frac{1000}{2+e^{-1000(t+c)}};$$

$$y = \frac{1000}{2+e^{-1000t}\cdot e^{-1000c}}, \quad e^{-1000c} = k$$

$$y = \frac{1000}{2+k\cdot e^{-1000t}} \quad \text{general solution;}$$

$$y(0) = 100$$

$$\frac{1000}{2+k\cdot 1} = 100 \quad ; \; k = 8$$

$$y(t) = \frac{1000}{2+8e^{-1000t}} = \frac{500}{1+4\cdot e^{-1000t}}$$

$t \to +\infty$;

$y(t) \to 500$;

500000 ~~[illegible]~~ is max number of species

#2 Linear equation (and integration factor)

\# Linear first order diff. equation is one of the form

$$\frac{dy}{dx} + y\,P(x) = Q(x)$$

$$\mu(x): \quad \frac{d}{dx}\left(\mu(x)\cdot y\right) = \frac{dy}{dx}\mu(x) + y\,P(x)\mu(x)$$

EXAMPLE: $$\frac{dy}{dx} + xy = x \quad \Big/ \; e^{\frac{x^2}{2}}$$

EXAMPLE:

$$\frac{dy}{dx} + xy = x \quad / \, e^{\frac{x^2}{2}}$$

$$\underline{\frac{dy}{dx} \cdot e^{x^2/2} + x e^{x^2/2} \cdot y} = x \cdot e^{x^2/2}$$

$$\frac{d}{dx}\left(y \cdot e^{x^2/2}\right) = \underline{\frac{dy}{dx} \cdot e^{x^2/2} + y \cdot e^{x^2/2} \cdot 2x}$$

$$\frac{d}{dx}\left(y \cdot e^{x^2/2}\right) = x$$

$$y \cdot e^{x^2/2} = \int x \, dx$$

$$y e^{x^2/2} = x + C \quad / \, e^{-x^2/2}$$

$$y = (x+C) \cdot e^{-x^2/2} \checkmark$$

Mistery: How to find $\mu(x)$?

Answer: $\boxed{\mu(x) = e^{\int P(x)\,dx}}$

$$y' + P(x) \cdot y = Q(x)$$

EXAMPLE: $x \cdot \frac{dy}{dx} + 2y = x^2 \quad / : x$

$$\boxed{\frac{dy}{dx} + \frac{2}{x} \cdot y = x \checkmark \quad (\mathcal{L}.\mathcal{E})}$$

$$P(x) = \frac{2}{x}$$

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2}$$

$$\mu(x) = x^2 \checkmark$$

$$(\mu(x) \cdot y)' = \mu(x) Q(x)$$

$$(x^2 \cdot y)' = x^2 \cdot x$$

$$x^2 y = \int x^3 dy$$

$$x^2 y = \int x^3 dy$$

$$x^2 y = \frac{x^4}{4} + c \quad / : x^2$$

$$y = \frac{x^2}{4} + \frac{c}{x^2} \quad \text{General solution}$$

- $x = 1 \Rightarrow y = 3$ ✓

$$y(1) = \frac{1^2}{4} + \frac{c}{1^2} = 3 \Rightarrow c + \frac{1}{4} = 3$$

$$c = 3 - \frac{1}{4} = \frac{12-1}{4} = \frac{11}{4};$$

$$y = \frac{x^2}{4} + \frac{\frac{11}{4}}{x^2} = \frac{x^2}{4} + \frac{11}{4x^2} = \frac{1}{4} \cdot \left(x^2 + \frac{11}{x^2}\right)$$

\# (1)

$$(x+1) \frac{dy}{dx} = e^x - 2y$$

$$\boxed{y' + P(x) \cdot y = Q(x}$$

$$(x+1) \cdot y' = e^x - 2y$$

$$(x+1) \cdot y' + 2y = e^x \quad / : (x+1)$$

$$y' + \frac{2}{x+1} y = \frac{e^x}{x+1} \quad ✓$$

$$\mu(x) = e^{\int \frac{2}{x+1} dx} \quad P(x) = \frac{2}{x+1}$$

$$\mu(x) = e^{2 \ln(x+1)} = e^{\ln(x+1)^2}$$

$$\mu(x) = (x+1)^2$$

$$(\mu(x) \cdot y)' = \mu(x) \cdot Q(x) = (x+1)^{\cancel{2}} \cdot \frac{e^x}{\cancel{x+1}}$$

$$\left((x+1)^2 y\right)' = (x+1) e^x$$

$$\left((x+1)^2 \cdot y\right)' = (x+1)e^x \quad / \int$$

$$(x+1)^2 \cdot y = \int (x+1) \cdot e^x \, dx$$

$$u = x+1 \quad , \quad e^x dx = dv$$
$$du = dx \quad ; \quad e^x = v$$

$$(x+1)^2 \cdot y = (x+1) \cdot e^x - \int e^x dx$$

$$(x+1)^2 \cdot y = (x+1) \cdot e^x - e^x + C$$

$$(x+1)^2 \cdot y = e^x \cdot (x+1-1) + C$$

$$(x+1)^2 \cdot y = x \cdot e^x + C \quad / : (x+1)^2$$

$$y = \frac{x}{(x+1)^2} \cdot e^x + \frac{C}{(x+1)^2} \quad \checkmark$$

General solution;

*(2) $$x y^2 \frac{dy}{dx} - x^2 \cdot e^y = e^y$$

$$x y^2 \cdot y' - x^2 e^y = e^y$$

$\boxed{\text{calculation for } y'}$

$$x y^2 \cdot y' = x^2 e^y + e^y = (x^2+1) e^y$$

$$y' = \frac{(x^2+1) e^y}{x y^2} = \frac{x^2+1}{x} \cdot \frac{e^y}{y^2}$$

Saparable equation!

$$\frac{dy}{dx} = \frac{x^2+1}{x} \cdot \frac{e^y}{y^2}$$

$$\frac{y^2 dy}{e^y} = \frac{x^2+1}{x} \cdot dx \quad \int$$

$$\frac{y^2\, dy}{e^y} = \frac{x^2+1}{x} \cdot dx$$

$$\int \frac{y^2}{e^y}\, dy = \int \frac{x^2+1}{x}\, dx$$

$$\int y^2 \cdot e^{-y}\, dy = \int \left(x + \frac{1}{x}\right) dx$$

(partial integration) $= \frac{x^2}{2} + \ln x + c$

Second order equations

"width constant coefficents"

(1) $a y'' + b y' + c y = f(x)$

$$\left[a \cdot \frac{d^2 y}{dx^2} + b \cdot \frac{dy}{dx} + c \cdot y = f(x) \right]$$

Auxiliary equation:

$a \cdot \lambda^2 + b\lambda + c = 0$; $ay'' + by' + cy = 0$

$\lambda \in \mathbb{C}$

Homogeneous equation

Algorithm:

(1) Solve homogenious part:
$ay'' + by' + cy = 0 \rightarrow y_h$

(2) Find particular solution for:
$ay'' + by' + cy = f(x) \rightarrow y_p$

(3) Rule (theorem):
General solution: $y = y_h + y_p$

(I) $ay'' + by' + cy = 0$

Auxiliary equation:

$$a\lambda^2 + b\lambda + c = 0, \quad \lambda \in \mathbb{C}$$

(i) $\lambda_1 \neq \lambda_2$; $\lambda_1, \lambda_2 \in \mathbb{R}$

$$y_h = c_1 \cdot e^{\lambda_1 x} + c_2 \cdot e^{\lambda_2 x}$$

(ii) $\lambda_1 = \lambda_2 \in \mathbb{R}$

$$y_h = (c_1 x + c_2) e^{\lambda_1 x}$$

(iii) $\lambda_{1,2} = \alpha \pm i \cdot \beta$; $i^2 = -1$, $\sqrt{-1} = i$

$$y_h = e^{\alpha x} \left(c_1 \cos(\beta x) + c_2 \sin(\beta x) \right)$$

(II) Finding particular solutions:

$$a y'' + b y' + c y = f(x)$$

rules depend on forms of; $f(x)$

$$f(x) = ax + b = 2x + 3$$

$$y_p = Ax + B$$

EXAMPLE: $\frac{dy^2}{}$

$$y(0) = 3, \quad y'(0) = -1;$$

$$y'' - y' - 2y = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \to -1, \; 2$$

$\lambda_1, \lambda_2 \in \mathbb{R}$ $\quad \lambda_1 \neq \lambda_2$

$\lambda_1, \lambda_2 \in \mathbb{R} \quad \lambda_1 \neq \lambda_2$

$y_h = c_1 \cdot e^{-x} + c_2 \cdot e^{2x}$;

(ii) $y'' - y' - 2y = x^2$

$y_p = \underline{a}x^2 + \underline{b}x + \underline{c}$

$y_p' = 2ax + b$

$y_p'' = 2a$

$y_p'' - y_p' - 2y = x^2$

$2a - (2ax + b) - 2\cdot(ax^2 + bx + c) = x^2$

$-2a \cdot x^2 + (-2a - 2b)\cdot x + (2a - b - 2c) = x^2$

$$\left.\begin{array}{ll} x^2: & -2a = 1 \\ x^1: & -2a - 2b = 0 \\ x^0: & 2a - b - 2c = 0 \end{array}\right\} \begin{array}{l} \to \underline{a} = -\frac{1}{2} \\ \underline{b} = -a = \frac{1}{2} \\ -1 - \frac{1}{2} - 2c = 0 \\ \quad 2c = -\frac{3}{2} \Rightarrow \underline{c} = -\frac{3}{4} \end{array}$$

$y_p = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$

$\therefore \quad y = y_h + y_p = c_1 e^{-x} + c_2 e^{2x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$

(ii) $y(0) = 3$, $y'(0) = -1$;

$y(0) = c_1 \cdot e^{-0} + c_2 \cdot e^{2\cdot 0} - \frac{3}{4} = 3$

$\boxed{c_1 + c_2 = \frac{15}{4}}$ ✓

$y' = c_1 \cdot e^{-x}\cdot(-1) + c_2 \cdot e^{2x}\cdot 2 - x + \frac{1}{2}$

$y'(0) = -c_1 + 2c_2 - \frac{1}{2} = -1$ ✓

$\boxed{-c_1 + 2c_2 = -\frac{1}{2}}$

$$\left.\begin{aligned} c_1 + c_2 &= \frac{15}{4} \\ -c_1 + 2c_2 &= -1 \end{aligned}\right\} +$$

$$3c_2 = \frac{11}{4} \quad \Rightarrow \quad c_2 = \frac{11}{12}$$

$$c_1 + c_2 = \frac{15}{4} \quad \Rightarrow \quad c_1 + \frac{11}{12} = \frac{15}{4}$$

$$c_1 = \frac{15}{4} - \frac{11}{12} = \frac{45-11}{12} = \frac{34}{12}$$

$$c_1 = \frac{17}{6} \checkmark$$

$$\boxed{y = \frac{17}{6} \cdot e^{-x} + \frac{11}{12} \cdot e^{2x} - \frac{1}{2} x^2 + \frac{1}{2} x - \frac{3}{4}}$$

EXAMPLE: $\frac{dy^2}{dx^2} - 2 \cdot \frac{dy}{dx} + 10y = 0$

(i) $y(0) = 1, \quad y(\frac{\pi}{6}) = 0$

(ii) Determine the behavior of the solution

(i) $y'' - 2y' + 10y = 0$

$$\lambda^2 - 2\lambda + 10 = 0$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm \sqrt{-36}}{2}$$

6i

$$\lambda_{1,2} = \frac{2 \pm \sqrt{36 \cdot (-1)}}{2} = \frac{2 \pm 6 \cdot i}{2} = 1 \pm 3 \cdot i$$

$$\alpha = 1, \quad \beta = 3$$

$$y = e^{x} \cdot (c_1 \cos(3x) + c_2 \sin(3x))$$

$y(0) = 1$: $\quad e^0 (c_1 \cos 0 + c_2 \sin 0) = 1$

$$c_1 = 1 \checkmark$$

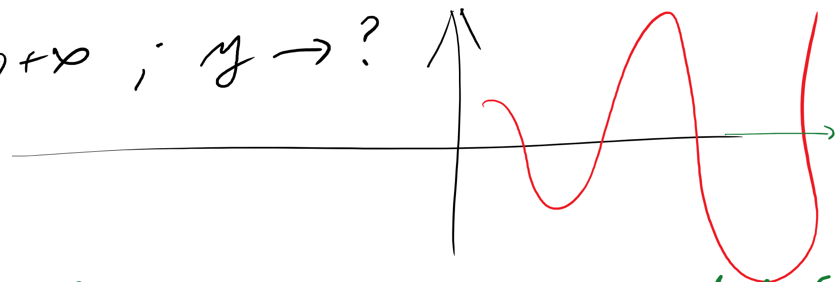
$y(\frac{\pi}{6}) = 0$: $\quad e^{\frac{\pi}{6}} \cdot (c_1 \cos \frac{\pi}{2} + c_2 \sin \frac{\pi}{2}) = 0$

$c_2 = 0$ ✓

$y = e^{x} \cdot \cos(3x)$

(ii) Behaviour of solution?

$x \to +\infty \; ; \; y \to ?$



- oscillate width unbounded amplitude!

• EXAMPLE: Find general solution:

$$\frac{d^2y}{dx^2} + 9y = x \cdot e^{x}$$

• $y'' + 9y = 0$

$\lambda^2 + 9 = 0 ; \quad \lambda^2 = -9$

$\lambda = \pm\sqrt{-9} = \pm 3 \cdot i$

3° $\alpha = 0 ; \; \beta = 3$

$y_a = c_1 \cos(3x) + c_2 \cdot \sin(3x)$

•• $y'' + 9y = x \cdot e^{x}$

$y_p = (ax + b) \cdot e^{x}$ ✓

$y_p' = a \cdot e^{x} + (ax + b) \cdot e^{x}$

$y_p' = (ax + a + b) \cdot e^{x}$ ✓

$y_p'' = a \cdot e^{x} + (ax + a + b) \cdot e^{x}$

$y_p'' = (ax + 2a + b) \quad e^{x}$ ✓

$y_p'' + 9y_p = x \cdot e^{x}$

$$(ax+2a+b)\cdot e^x + 9\cdot(ax+b)e^x = x\cdot e^x \quad / : e^x$$

$$ax + 2a + b + 9ax + 9b = x$$

$$10ax + 2a + 10b = x$$

x^1: $10a = 1 \implies a = \frac{1}{10}$ ✓

x^0: $2a + 10b = 0$

$$10b = -2a$$

$$b = -\frac{1}{5}a = -\frac{1}{5}\cdot\frac{1}{10} = -\frac{1}{50}$$ ✓

$$y_p = \left(\frac{1}{10}x - \frac{1}{50}\right)\cdot e^x = \frac{1}{50}\cdot(5x-1)\cdot e^x$$

(∴) $y = y_h + y_p$

$$= c_1\cos(3x) + c_2\sin(3x) + \frac{1}{50}\cdot(5x-1)\cdot e^x$$ ✓

Behavior of solutions

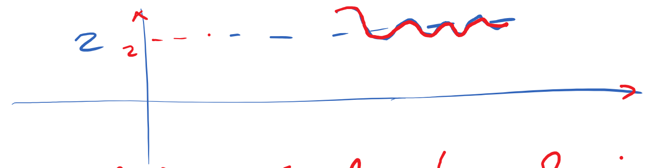
(i) Describe long term behavior of $y(t)$?

(ii) How solution evolves as x tends to infinity?

EXAMPLES:

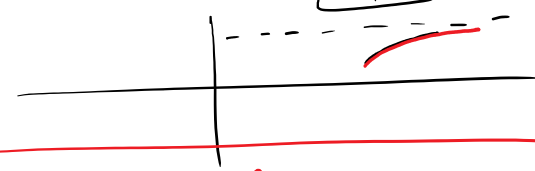
(1) $p(t) = 2 + e^{-t}\cdot\cos(2t)$

(1) $p(t) = 2 + e^{-t} \cdot \cos(2t)$



oscillatory tends to 2;

(2) $p(t) = 2 - \frac{1}{16t+1} \xrightarrow[t \to +\infty]{} 2$

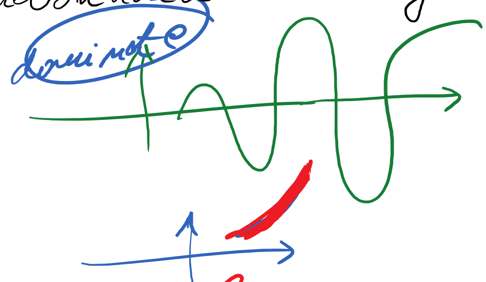


increasingly tends to 2

(III) $y(x) = e^{2x} + e^{4x} \cdot (5\cos(3x) + 4\sin(3x))$

oscillate width unbounded increasing magnitude

dominate



IV $p(t) = 2 - e^{2t} + e^{3t}$

increase unboundedly