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Coupled differential equations

Motivation:

Population growth
logistic model

- two species
  populations: $y_1(t)$, $y_2(t)$

$$\begin{cases} \frac{1}{y_1} \cdot \frac{dy_1}{dt} = a_1 - b_1 y_1 - c_1 y_2 \\ \frac{1}{y_2} \cdot \frac{dy_2}{dt} = a_2 - b_2 y_2 - c_2 y_1 \end{cases}, \quad a_1, b_1, c_1, a_2, b_2, c_2 > 0$$

$$\begin{cases} \frac{dy_1}{dt} = a_1 y_1 - b_1 y_1^2 - c_1 y_1 y_2 \\ \frac{dy_2}{dt} = a_2 y_2 - b_2 y_2^2 - c_2 y_1 y_2 \end{cases}$$

In General:

$$\begin{cases} \frac{dy_1}{dt} = f_1(y_1, y_2, \dots, y_n, t) \\ \frac{dy_2}{dt} = f_2(y_1, y_2, \dots, y_n, t) \\ \vdots \\ \frac{dy_n}{dt} = f_n(y_1, y_2, \dots, y_n, t) \end{cases}$$

System of differential equations

$n=2$ $\quad f_i$ is linar by $y_1, y_2, \dots, y_n$

Reducing to a second order equations

Example: $y_1(t)$, $y_2(t)$; $y_1(0) = 5$ $y_2(0) = 2$

Example: $y_1(t),\ y_2(t);\quad y_1(0)=5\quad y_2(0)=-2$

$$\begin{cases} (1)\ \dfrac{dy_1}{dt} = 2y_1 + 4y_2 \\ (2)\ \dfrac{dy_2}{dt} = 3y_1 + 3y_2 \end{cases}$$

$$(1) \Longrightarrow \quad \frac{d^2y_1}{dt^2} = 2\cdot\frac{dy_1}{dt} + 4\cdot\frac{dy_2}{dt}$$

$$\frac{d^2y_1}{dt^2} = 2\cdot\frac{dy_1}{dt} + 4\cdot(3y_1 + 3y_2)$$

$$\frac{d^2y_1}{dt^2} = 2\cdot\frac{dy_1}{dt} + 12y_1 + 3\cdot 4y_2$$

$$(1)\quad 4y_2 = \frac{dy_1}{dt} - 2y_1$$

$$* \quad \frac{d^2y_1}{dt^2} = 2\cdot\frac{dy_1}{dt} + 12y_1 + 3\cdot\left(\frac{dy_1}{dt} - 2y_1\right)$$

$$\frac{d^2y_1}{dt^2} = 5\cdot\frac{dy_1}{dt} + 6y_1$$

$$* \quad \frac{d^2y_1}{dt^2} - 5\cdot\frac{dy_1}{dt} - 6y_1 = 0$$

(second order diff. equation)

$$\lambda^2 - 5\lambda - 6 = 0 \quad \text{(auxiliary equation)}$$

$$\lambda_{12} = \frac{5 \pm \sqrt{25+24}}{2} = \frac{5\pm 7}{2} \to \begin{matrix} -1 \\ 6 \end{matrix}$$

$$y_1(t) = c_1\cdot e^{-t} + c_2\cdot e^{6t}$$

$$(1)\quad \frac{dy_1}{dt} = 2y_1 + 4y_2$$

$$\frac{dy_1}{dt} = -c_1\cdot e^{-t} + 6c_2\cdot e^{6t}$$

$$-c_1\cdot e^{-t} + 6c_2 e^{6t} = 2c_1 e^{-t} + 2c_2 e^{6t} + 4y_2$$

$$4y_2 = -3c_1 e^{-t} + 4c_2\cdot e^{6t}$$

$$\begin{cases} y_2 = -\frac{3}{4}c_1\cdot e^{-t} + c_2\cdot e^{6t} \\ y_1 = c_1\cdot e^{-t} + c_2\cdot e^{6t} \end{cases} \quad \text{General solution}$$

$$y_1(0)=5\ ;\ y_2(0)=-2$$

$$y_1(0) = c_1 + c_2 = 5$$
$$y_2(0) = -\tfrac{3}{4}c_1 + c_2 = -2 \quad \cdot(-1)$$

$$\tfrac{7}{4}c_1 = 7 \Rightarrow c_1 = 4 \qquad c_2 = 1$$

$$\boxed{\begin{aligned} y_1 &= 4\cdot e^{-t} + e^{6t} \\ y_2 &= -3e^{-t} + e^{6t} \end{aligned}}$$

EXAMPLE: $y(t),\ z(t)$

$$\begin{cases} \dfrac{dy}{dt} = -4y + 2z \\ \dfrac{dz}{dt} = -2y + 4t^2 + 4 \end{cases} ; \qquad y(0) = 1,\quad z(0) = \tfrac{7}{2}$$

$$\frac{d^2y}{dt^2} = -4\cdot\frac{dy}{dt} + 2\cdot\frac{dz}{dt}$$

$$\frac{d^2y}{dt^2} = -4\cdot\frac{dy}{dt} + 2\cdot(-2y + 4t^2 + 4)$$

$$\frac{d^2y}{dt^2} = -4\cdot\frac{dy}{dt} - 4y + 8t^2 + 8$$

$$\boxed{\frac{d^2y}{dt^2} + 4\cdot\frac{dy}{dt} + 4y = 8t^2 + 8}$$

$$\frac{dy^2}{dt^2} + 4\cdot\frac{dy}{dt} + 4y = 0$$

$$\lambda^2 + 4\lambda + 4 = 0 \iff (\lambda+2)^2 = 0$$

$$\lambda_1 = \lambda_2 = -2$$

$$y_h(t) = (c_1 t + c_2)\cdot e^{-2t}$$

$$\begin{cases} y_p(t) = at^2 + bt + c \\ y_p'(t) = 2at + b, \quad y_p'' = 2a \end{cases}$$

$$\boxed{y'' + 4y' + 4 = 8t^2 + 8}$$

$$\ldots + 4\cdot(at^2 + bt + c) = 8t^2 + 8$$

$$2a + 4\cdot(2at+b) + 4\cdot(at^2+bt+c) = 8t^2+8$$

$$4at^2 + (8a+4b)t + 2a+4b+4c = 8t^2+8$$

t^2: $4a = 8 \Rightarrow a=2$

t^1: $8a+4b=0 \Rightarrow b=-4$

t^0: $2a+4b+4c = 8$

$\cancel{8} - 16 + 4c = \cancel{8}$ $\quad c=4$

$$y_p(t) = 2t^2-4t+4$$

$$y(t) = y_h(t) + y_p(t)$$

$$\boxed{y(t) = (c_1t+c_2)\cdot e^{-2t} + 2t^2-4t+4}$$

(1) $\quad \frac{dy}{dt} = -4y + 2z$

$$\frac{dy}{dt} = c_1\cdot e^{-2t} + (c_1t+c_2)\cdot e^{-2t}\cdot(-2) + 4t-4$$

$$\frac{dy}{dt} = (-2c_1t + c_1-2c_2)\cdot e^{-2t} + 4t-4 \checkmark$$

$$(-2c_1t + c_1-2c_2)\cdot e^{-2t} + 4t-4 = -4\cdot\left[(c_1t+c_2)\cdot e^{-2t} + 2t^2-4t+4\right] + 2z$$

$$2z = 4\cdot(c_1t+c_2)\cdot e^{-2t} + 8t^2-16t+16 + (-2c_1t+c_1-2c_2)e^{-2t} + 4t-4$$

$$2z = e^{-2t}\cdot(4c_1t + 4c_2 - 2c_1t + c_1 - 2c_2) + 8t^2-12t+12$$

$$2z = (2c_1t + c_1+2c_2)\cdot e^{-2t} + 8t^2-12t+12$$

$$\boxed{z = (c_1t + c_2 + \tfrac{1}{2}c_1)\cdot e^{-2t} + 4t^2-6t+6}$$

$$\left.\begin{aligned} z(0) &= c_2 + \tfrac{1}{2}c_1 + 6 = \tfrac{7}{2} \\ y(0) &= c_2 + 4 = 1 \end{aligned}\right\} \quad \boxed{c_2 = -3} \checkmark$$

$-3+\frac{1}{2}$

$-\frac{5}{2}$

$$-3 + \tfrac{1}{2}c_1 + 6 = \tfrac{7}{2}$$

$$\tfrac{1}{2}c_1 = -3 + \tfrac{7}{2} = \tfrac{1}{2} \quad \boxed{c_1 = 1} \checkmark$$

Solution:

$$\left\{\begin{aligned} y(t) &= (t-3)\cdot e^{-2t} + 2t^2-4t+4 \\ z(t) &= (t-\tfrac{5}{2})\cdot e^{-2t} + 4t^2-6t+6 \end{aligned}\right\}$$

$\boxed{\#\#\#3}$ Using diagonalisation;

$$\frac{dy}{dt} = Ay$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \frac{dy}{dt} = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix}$$

A type n times n

$$\frac{dy}{dt} = Ay + G(t)$$

$n=2$
$$\frac{dy}{dt} = A \cdot y + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

A can be diagonised
there is invertible matrix P and diagonal matrix D, such that

$$P^{-1}AP = D$$

$\boxed{\text{Motiv:}}$
$$\begin{cases} y'(t) = a\,y(t) \\ z'(t) = b\,z(t) \end{cases}$$

$$\frac{dy}{dt} = a \cdot y$$

$$\int \frac{dy}{y} = \int a \cdot dt$$

$$\ln y = a \cdot t + c_1$$

$$y = e^{at + c_1}$$

$$y = e^{c_1} \cdot e^{at}$$

$$\boxed{\begin{aligned} y &= A \cdot e^{at} \\ z &= B \cdot e^{bt} \end{aligned}}$$

$\boxed{\text{substitution:}}$ $y = Pz$ $\quad P^{-1}AP = D$

$$\frac{dy}{dt} = \frac{d}{dt}(Pz) = P \cdot \frac{dz}{dt}$$

$$\frac{dy}{dt} = \frac{d}{dt}(Pz) = P\cdot\frac{dz}{dt} \checkmark$$

$$\boxed{y' = Ay}$$

$$P\cdot\frac{dz}{dt} = A\cdot Pz \quad \Big/ \cdot P^{-1} \text{ from left sid}$$

$$P^{-1}P\cdot\frac{dz}{dt} = P^{-1}APz$$

$$\boxed{\frac{dz}{dt} = D\cdot z}$$

EXAMPLE:

$$\frac{dy_1}{dt} = 2y_1 + 4y_2; \quad y_1(0) = 5$$

$$\frac{dy_2}{dt} = 3y_1 + 3y_2; \quad y_2(0) = -2$$

Matrix form: $\frac{dy}{dt} = A\cdot y$; $\left\{ \frac{dy}{dt} = \begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right.$

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}$$

(i) eigen values

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

eigen value $\lambda_1 \rightarrow$ eigen vector$_1$ ' e_1

eigen value $\lambda_2 \rightarrow$ eigen vector$_2$ e_2

$$P = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

$$\det(A - \lambda E) = 0$$

$$\det(A-\lambda E)=0$$

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{vmatrix}=0$$

$$(2-\lambda)(3-\lambda)-12=0$$

$$\lambda^2-5\lambda+6-12=0$$

$$\lambda^2-5\lambda-6=0$$

$$\lambda_{1,2}=\frac{5\pm\sqrt{25+24}}{2}=\frac{5\pm7}{2} \begin{array}{l} \nearrow -1 \\ \to 6 \end{array}$$

$$D=\begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix} \checkmark$$

$\boxed{\text{eigen vectors}}$

$$(A-\lambda_1 E)\cdot x=0$$

$\lambda_1=-1$

$$\left(\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}-(-1)\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$$

$$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$$

$$\begin{cases} 3x_1+4x_2=0; \\ 3x_1+4x_2=0 \end{cases} \checkmark$$

$$x_2=\alpha$$

$$3x_1+4\alpha=0$$

$$3x_1=-4\alpha$$

$$x_1=-\tfrac{4}{3}\alpha$$

$$V_1=\left\{\begin{pmatrix} -4/3\,\alpha \\ \alpha \end{pmatrix} \middle| \alpha\in\mathbb{R}\right\}$$

$$\begin{pmatrix} -\frac{4}{3}\alpha \\ \alpha \end{pmatrix}=\alpha\cdot\begin{pmatrix} -4/3 \\ 1 \end{pmatrix}$$

e_1' $\qquad$ $\boxed{\alpha=1}$

$\alpha\neq0!$ $\qquad$ $\alpha=3$ $\Rightarrow$ $\boxed{e_1=\begin{pmatrix} -4 \\ 3 \end{pmatrix}}$

λ_2: $\quad (A-\lambda E)x=0$

$$(A-6E)x=0$$

$$\left(\begin{pmatrix} 2 & 4 \\ \end{pmatrix}-6\cdot\begin{pmatrix} 1 & 0 \\ \end{pmatrix}\right)\begin{pmatrix} x_1 \\ \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A-6E)x=0$$

$$\left(\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} - 6\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} -4x+4y=0 \\ 3x-3y=0 \end{cases} \Longleftrightarrow \quad x-y=0$$

$$\boxed{y=x} =: \alpha$$

$$V_2 = \left\{ \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \middle| \alpha \in \mathbb{R} \right\}$$

$$\alpha = 1 \Rightarrow e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} e_1 & e_2 \end{pmatrix} = \begin{pmatrix} -4 & 1 \\ 3 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}$$

$$y' = A\cdot y \quad , \quad y = Pz \text{ (substitution)}$$

$$y' = P\cdot z'$$

$$Pz' = A\cdot Pz \quad \Big| \cdot P^{-1}$$

$$P^{-1}Pz' = P^{-1}APz$$

$$z' = Dz$$

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$z_1' = -z_1 \Rightarrow z_1 = c_1 e^{-t}$$

$$z_2' = 6z_2 \Rightarrow z_2 = c_2 e^{6t}$$

transforming initial conditions

$$y_1(0)=5$$
$$y_2(0)=-2 \quad ; \quad y=Pz$$

$$y=Pz \quad / \; P^{-1}$$
$$P^{-1}y=P^{-1}P\cdot z$$
$$z=P^{-1}\cdot y$$
$$\begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} = P^{-1}\cdot\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}$$

$$P=\begin{pmatrix} -4 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\det P=-4-3=-7\neq 0 \checkmark$$

$$P^{-1}=\frac{1}{\det P}\cdot \operatorname{adj} P=\frac{1}{-7}\begin{pmatrix} 1 & -3 \\ -1 & -4 \end{pmatrix}^T$$

$$P^{-1}=-\frac{1}{7}\cdot\begin{pmatrix} 1 & -1 \\ -3 & -4 \end{pmatrix}=\frac{1}{7}\begin{pmatrix} -1 & 1 \\ 3 & 4 \end{pmatrix} \checkmark$$

$$\begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}=\frac{1}{7}\begin{pmatrix} -1 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix}=\frac{1}{7}\begin{pmatrix} -7 \\ 7 \end{pmatrix}=\begin{pmatrix} -1 \\ 1 \end{pmatrix} \checkmark$$

$$z_1(0)=-1 \qquad z_1(t)=c_1\cdot e^{-t}$$
$$z_2(0)=1 \quad ; \qquad z_2(t)=c_2\cdot e^{6t}$$

$$z_1(0)=c_1=-1$$
$$z_2(0)=c_2=1$$

$$\left\{ \begin{aligned} z_1(t)&=-e^{-t} \\ z_2(t)&=e^{6t} \end{aligned} \right\}$$

$$y=Pz=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} -4 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -e^{-t} \\ e^{6t} \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} 4e^{-t}+e^{6t} \\ -3e^{-t}+e^{6t} \end{pmatrix}$$

$$\left\{ y_1(t)=e^{6t}+4e^{-t} \right.$$

$$\left\{\begin{array}{l} y_1(t) = e^{6t} + 4e^{-t} \\ y_2(t) = e^{6t} - 3e^{-t} \end{array}\right\}$$

Question 5.10

$$\left\{\begin{array}{l} \frac{df}{dt} = 3f(t) - g(t) + \dots \\ \frac{dg}{dt} = 3g(t) - f(t) \\ f(0) = 2, \quad g(0) = 0 \end{array}\right\}$$

- $x = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$

$$x' = Ax \quad , \quad A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

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substitution: $x = Py$; $\boxed{P^{-1}AP = D}$

$x' = P \cdot y'$

$Py' = AP \cdot y \quad / P^{-1}$

$y' = Dy$ ✓

eigenvalues:

$$\det(A - \lambda E) = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 - 1 = 0$$

$$(3-\lambda)^2 = 1$$

$$3-\lambda = 1 \quad \vee \quad 3-\lambda = -1$$

$$\lambda_1 = 2 \quad \vee \quad \lambda_2 = 4$$

eigenvectors:

$\lambda_1 = 2$: $\quad (A - 2E)e = 0$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = 0 \quad \left\{\begin{array}{l} e_1 - e_2 = 0 \\ -e_1 + e_2 = 0 \end{array}\right. \Longleftrightarrow \quad e_1 - e_2 = 0$$

$$e_1 = e_2 = \alpha \quad , \quad \alpha = 1 \checkmark$$

$$P_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \checkmark$$

$$\lambda_2 = 4 : \quad (A - 4E) \cdot e = 0$$

$$\begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -e_1 - e_2 &= 0 \\ 3e_1 + 3e_2 &= 0 \end{aligned}\right\} \quad \begin{aligned} &-e_1 - e_2 = 0 \\ &e_1 = \alpha \ ; \ e_2 = -\alpha \end{aligned}$$

$$\alpha = 1 : \quad P_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

$$y' = D \cdot y \ ; \quad \left.\begin{aligned} y_1 &= c_1 \cdot e^{2t} \\ y_2 &= c_2 \cdot e^{4t} \end{aligned}\right\}$$

$$x = P \cdot y \iff \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \cdot e^{2t} \\ c_2 \cdot e^{4t} \end{pmatrix}$$

$$\begin{cases} f(t) = c_1 \cdot e^{2t} + c_2 \cdot e^{4t} \\ g(t) = c_1 \cdot e^{2t} - c_2 \cdot e^{4t} \end{cases}$$

initial conditions:

$$\left.\begin{aligned} f(0) &= c_1 + c_2 = 2 \\ g(0) &= c_1 - c_2 = 0 \end{aligned}\right\} \qquad \begin{aligned} 2c_1 &= 2 \\ c_1 = 1 \quad & c_2 = 1 \end{aligned}$$

$$\boxed{\begin{aligned} f(t) &= e^{2t} + e^{4t} \\ g(t) &= e^{2t} - e^{4t} \end{aligned}}$$

Solution

#3 Applications of diff. equations

PAUSE : $\dfrac{13^{20}}{13^{40}}$

Macroeconomics

Investment - $I(t)$
Production - $Q(t)$
income - $Y(t)$
consumption - $C(t)$

equilibrium conditions:

$$\begin{cases} Q(t) = Y(t) \\ Y(t) = C(t) + I(t) \end{cases}$$

hypothesis:

1 $\boxed{C(t) = c + b \cdot Y(t)}$

$$\frac{dC}{dt} = b \cdot \frac{dY}{dt}$$

$$\frac{dC}{dY} = b$$ – marginal propensity to consume

$$Y = C + I$$

$$I = Y - C = Y - c - b \cdot Y$$

$$I = -c + s \cdot Y;$$

$$\frac{dI}{dt} = s \cdot \frac{dY}{dt}$$

$$\frac{dI}{dY} = s$$ – marginal propensity to invest

2 - production increase at a rate proportional to investment

$$\frac{dQ}{dt} = \rho \cdot I \; ; \quad \boxed{\rho - \text{consant}}$$

$$\frac{dI}{dt} = \frac{dI}{dY} \cdot \frac{dY}{dt} = s \cdot \frac{dY}{dt} = s \cdot \frac{dQ}{dt} \overset{2}{=} s \cdot \rho \cdot I$$

(diff. equation)

$$\frac{dI}{\ } \quad s \cdot \rho \cdot I$$

$$\frac{\ }{dt} = s\cdot\varrho\cdot$$

$$\frac{dI}{I} = s\cdot\varrho\cdot dt$$

$$\ln I = s\cdot\varrho\cdot t + c$$

$$I = e^{s\cdot\varrho\cdot t + c} = e^{c}\cdot e^{s\cdot\varrho\cdot t}$$

$$I(t) = c_1\cdot e^{s\cdot\varrho\cdot t}, \quad I(0) = c_1;$$

$$\boxed{I(t) = I(0)\cdot e^{s\cdot\varrho\cdot t}} \quad ✓✓$$

Continuous cash flow

$A(t)$ - balance

(1) • rate of increase is proportional to the amount

(**) $$\frac{\frac{dA}{dt}}{A} = r \quad ; \boxed{r - \text{constant}}$$

$$A = A(0)\cdot e^{rt}$$

(2) money is continuously added to the account at rate $f(t)$

$$\frac{dA}{dt} = f(t) + r\cdot A(t)$$

$$\frac{dA}{dt} - r\cdot A(t) = f(t) \quad \text{— } \underline{\underline{\text{linear diff. eq.}}}$$

$\boxed{\text{EXAMPLE}}$ $f(t) = t$

$$\frac{dA}{dt} - r\cdot A(t) = t \quad \Big/ \mu(t)$$

integration factor : $P(t) = -r$

$$\mu(t) = e^{\int P(t)dt}$$

$$\mu(t) = e^{-\int r\cdot dt} = e^{-rt}$$

$$\frac{dA}{dt}\cdot\mu(t) - r\cdot\mu(t)\cdot A(t) = t\cdot\mu(t)$$

$$\frac{d}{dt}\left(A(t)\cdot\mu(t)\right) = t\cdot e^{-r\cdot t}$$

$$A(t)\cdot\mu(t) = \int t\cdot e^{-rt}\,dt$$

$$A(t)\cdot e^{-rt} =$$

$$u = t \qquad e^{-rt}\,dt = dv$$
$$du = dt \quad , \quad -\frac{1}{r}\cdot e^{-rt} = v$$

$$A(t)\cdot e^{-rt} = -\frac{t}{r}\cdot e^{-rt} - \int -\frac{1}{r}\cdot e^{-rt}\,dt$$

$$A(t)\cdot e^{-rt} = -\frac{t}{r}\cdot e^{-rt} + \frac{1}{r}\cdot\int e^{-rt}\,dt$$

$$A(t)\cdot e^{-rt} = -\frac{t}{r}e^{-rt} + \frac{1}{r}\cdot\frac{-1}{r}\cdot e^{-rt} + C \;\Big/\cdot e^{rt}$$

$$A(t) = \left(-\frac{1}{r^2} - \frac{t}{r}\right) + C\cdot e^{rt}$$
$$A(t) = C\cdot e^{rt} - \left(\frac{rt+1}{r^2}\right)$$

$$A(0) = C - \frac{1}{r^2} \Rightarrow C = A(0) + \frac{1}{r^2}$$

$$A(t) = \left(A(0) + \frac{1}{r^2}\right)\cdot e^{rt} - \frac{1+rt}{r^2}$$

$$A(t) = A(0)\cdot e^{rt} + \frac{e^{rt} - 1 - rt}{r^2}$$ ✓

Homework: Find $A(t)$ if $f(t) = 1$, for all t

#3 Continuous price adjustment

$p < p^*$

$q^D(p) > q^S(p)$

$x(p) = q^D(p) - q^S(p)$ - exsess of demand

assumption: rate of increase depend of excess of demand

p' - rate of increse price

$\boxed{p' = f(x(p))}$ - Saparalle diff. equation

EXAMPLE:

demand: $3q + 2p = 10$

supply: $q - 2p = -6$

price adjustment: $p' = (3 \cdot x(p))^3$

(1) $\begin{cases} 3q + 2p = 10 \\ q - 2p = -6 \end{cases}$ +

$4q = 4$

$q^* = 1$

$p^* = \frac{7}{2}$;

(2) $t = 0$; D: $3q + 2p = 10$

$3q = 10 - 2p$

$q = \frac{10 - 2p}{3}$

$q^D(p) = \frac{10 - 2p}{3}$

S: $q - 2p = -6$

$q = 2p - 6$

$q^S(p) = 2p - 6$

$x(p) = q^D(p) - q^S(p)$

$x(p) = \frac{10 - 2p}{3} - (2p - 6)$

$x(p) = \frac{10 - 2p - 6p + 18}{3} = \frac{28 - 8p}{3}$

$$\frac{dp}{dt} = f(x(p))$$

$$\frac{dp}{dt} = \left(3 \cdot x(p)\right)^3 = (28-8p)^3$$

Saparable diff eq.

$$\frac{dp}{(28-8p)^3} = dt$$

$$\int \frac{dp}{(28-8p)^3} = \int dt$$

$$-\frac{1}{2(28-8p)^2} \cdot -\frac{1}{8} = t + C$$

$$\int \frac{1}{x^3} dx = -\frac{1}{2x^2} + c$$

$$\int \frac{1}{x^7} dx = -\frac{1}{6x^6} + c$$

$$\frac{1}{16 \cdot (4 \cdot (7-2p))^2} = t + C$$

$$\frac{1}{256 \cdot (7-2p)^2} = t + C$$

$$\frac{1}{(7-2p)^2} = 256 \cdot (t+C)$$

$$\frac{1}{7-2p} = 16 \cdot \sqrt{t+C} \quad \vee \quad \frac{1}{7-2p} = -16\sqrt{t+C}$$

$$7-2p = \frac{1}{16\sqrt{t+C}} \quad \vee \quad 7-2p = -\frac{1}{16\sqrt{t+C}}$$

$$2p = 7 - \frac{1}{16\sqrt{t+C}} \quad \vee \quad 2p = 7 + \frac{1}{16\sqrt{t+C}}$$

$$p = \frac{7}{2} - \frac{1}{8\sqrt{t+C}} \quad \vee \quad p = \frac{7}{2} + \frac{1}{8\sqrt{t+C}}$$

$p^* = 7/2$ ✓ $\boxed{p < p^*}$ ✓

$$p(0) = 3 < p^* = \frac{7}{2}$$

$$p(t) = \frac{7}{2} - \frac{1}{8\sqrt{t+c}}$$

$$p(0) = \frac{7}{2} - \frac{1}{8\sqrt{0+c}} = 3$$

$$\frac{1}{8\sqrt{c}} = \frac{1}{2}$$

$$8\sqrt{c} = 2$$

$$\sqrt{c} = \frac{1}{4}$$

$$c = \frac{1}{16}$$

$$p(t) = \frac{7}{2} - \frac{1}{8\sqrt{t+\frac{1}{16}}}$$

$$p(t) = \frac{7}{2} - \frac{1}{8\sqrt{\frac{16t+1}{16}}} = \frac{7}{2} - \frac{1}{8 \cdot \frac{\sqrt{16t+1}}{4}}$$

$$p(t) = \frac{7}{2} - \frac{1}{2\sqrt{16t+1}} \quad \checkmark$$

$t \to +\infty$ $\qquad p(t) \to \frac{7}{2} = p^*$

increasingly

Market trends;

housing marke

consumers → try to anticipate trend

trend → brzina rasta

model:

trend

$$\begin{cases} (1) \quad q^D: \quad q^D = 9 - 6p + 5\frac{dp}{dt} - 2\cdot\frac{d^2p}{dt^2} \\ (2) \quad q^S: \quad q^S = -3 + 4p - \frac{dp}{dt^2} \end{cases}$$

$$q^D = q^S$$

$$9 - 6p + 5\cdot\frac{dp}{dt} - 2\frac{d^2p}{dt^2} = -3 + 4p - \frac{dp}{dt} - \frac{d^2p}{dt^2}$$

$$\frac{d^2p}{dt^2} - 6\cdot\frac{dp}{dt} + 10p = 12$$

Second order diff. equation

- $p'' - 6p' + 10 = 0$

$\lambda^2 - 6\cancel{p}\lambda + 10 = 0$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{36-40}}{2} = \frac{6 \pm 2i}{2} = 3 \pm i$$

$$p_h(t) = e^{3t}(c_1 \cos t + c_2 \sin t)$$

$p_p(t) = C \qquad p_p' = 0 \quad p_p'' = 0$

$10C = 12 \Rightarrow C = \frac{12}{10} = \frac{6}{5}$

$p(t) = p_h(t) + p_p(t) -$

$$p(t) = e^{3t}(c_1 \cos t + c_2 \sin t) + \frac{6}{5}$$

http:// aurora.etf.bg.ac.rs/~azdr